Abstract

Common folklore in growth theory suggests that the stability of balanced growth paths in the capital-labor space is essentially guaranteed by conditions which imply stability of the corresponding steady states of the models in intensity form. We show by means of simple examples that, in general, these well-known conditions are only necessary. For a class of deterministic growth models with discrete time we provide new structural insights into the nature of this phenomenon by stating additional requirements that ensure stability of balanced growth paths in the original capital-labor space. We introduce a notion of path-wise convergence for stochastic growth models and generalize our sufficient conditions to the stochastic case.
1 Introduction

Dynamic properties of growth models are usually analyzed in per-capita terms or in growth rates. The reason for doing so is the fact that growth models which take the original capital-labor space as state space are expanding and thus inherently unstable when the labor force grows. A concept of central importance for the analysis of the long-run development in growing economies is the notion of balanced growth paths along which the growth rate of capital is equal to the growth rate of the labor force. Reformulating such a model in per-capita terms, balanced growth paths are characterized by steady states of a model in intensity form. It is tacitly presumed that the stability of balanced growth paths, i.e., the convergence of nearby unbalanced growth paths to a balanced growth path is implied by the stability of the corresponding steady states of the model in intensity form. This assumption of ‘path-wise convergence’ underlies almost all models of economic growth, including overlapping-generations models, optimal growth models, and endogenous growth models.

A dynamic analysis of the growth model in the original state space is usually omitted. Indeed, the standard textbook literature, (e.g. Romer (1996), p. 14, Aghion & Howitt (1998), Chap. 1, or Barro & Sala-I-Martin 1995) seem to presume that the convergence in either per-capita terms or in growth rates implies the convergence in state space as well. However, the analysis typically lacks a rigorous mathematical notion of path-wise convergence in the capital-labor space.

Economically the issue of path-wise convergence is at the center of the debate of comparative growth as for example in Galor (1996), yet a rigorous analysis of the dynamics in the capital-labor space is missing. Notable exceptions are Deardorff (1970), Jensen (1994), Jensen, Alsholm, Larsen & Jensen (2005), and Pampel (2007). Deardorff (1970) shows that without depreciation the distance between unbalanced and balanced growth paths in the standard Solow model is always exploding. Jensen (1994) points out that, in general, path-wise convergence in continuous-time growth models cannot be obtained from convergence in per-capita quantities. Jensen, Alsholm, Larsen & Jensen (2005) investigates convergence in the state space for models with arbitrary elasticity to scale. There it is conjectured that models with depreciation will converge to the balanced growth path. The literature seems to have overlooked these findings. A rigorous analysis of continuous-time growth models is conducted by Pampel (2007) who demonstrates that two identical economies which converge in per-capita ratios but with arbitrarily small differences in initial capital may induce time series along which absolute differences in GDP, aggregate investment, or aggregate consumption diverge to infinity. These findings indicate that convergence in per-capita quantities is only a necessary condition for path-wise convergence and that one should be careful when interpreting convergence results on the level of per-capita quantities, growth rates, or logarithms as deviations from some general trend.

This paper provides a more structural insight into this phenomenon by providing necessary and sufficient conditions for path-wise convergence for a class of growth models in discrete time and with positive depreciation, both deterministic and stochastic. Based on the theory of random dynamical systems (Arnold 1998), the notion of path-wise convergence to a balanced growth path is generalized to stochastic models and sufficient conditions for convergence and divergence are provided, even for expanding systems. These results have implications for many dynamic models which exhibit convergence in intensity form only. Among these are models with overlapping generations, models of endogenous and of optimal growth, two sector models as initiated by Galor (1992) , and models in international trade, e.g., Mountford (1998, 1999).

The paper is organized as follows. Basic results on existence and uniqueness of balanced growth for the deterministic Solow model are collected in Section 2. Section 3 provides necessary and sufficient conditions for path-wise convergence in the state space for growth models with more general savings
functions which admit asymptotically stable fixed points in per-capita terms. Stochastic models with discrete time are analyzed in Section 4.

2 Balanced Growth

2.1 The basic model with exogenous savings

The basic issues of the relationship of the stability of balanced growth in the two alternative perspectives can best be exhibited and analyzed in the simplest of all growth models, the one originally discussed by Solow (1956). Let aggregate real income be generated by a continuous concave production function \( F : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) which is homogeneous of degree one. For given labor supply \( L \geq 0 \) and given capital stock \( K \geq 0 \), aggregate savings under full employment and full capital usage are given by a savings function

\[
S = S(L, K) := \bar{s}F(L, K), \quad (L, K) \in \mathbb{R}^2_+.
\]

where \( 0 \leq \bar{s} \leq 1 \) is a given constant savings propensity. Applying the fundamental principle of capital accumulation with a given rate of depreciation on capital \( 0 \leq \delta \leq 1 \), the law of capital accumulation is a mapping \( G_\delta : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) given by

\[
G_\delta(L, K) := (1 - \delta)K + S(L, K)
\]

which determines the capital stock of the subsequent period \( K_{t+1} = G_\delta(K_t, L_t) \) for any \( t = 0, 1, \ldots \). If the labor force grows at a constant rate \( n > -1 \) so that \( L_{t+1} = (1+n)L_t \), one obtains a dynamical system for \( L \) and \( K \) with state space \( \mathbb{R}^2_+ \) governed by the two maps

\[
(\mathcal{L}, G_\delta) : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+, \quad (L, K) \mapsto (\mathcal{L}(L, K), G_\delta(L, K)),
\]

where \( \mathcal{L}(L, K) := (1 + n)L \). Let \( (\mathcal{L}, G_\delta)^t \) denote the \( t \)-th iterate of the map \((\mathcal{L}, G_\delta)\), i.e.,

\[
(L_t, K_t) = (\mathcal{L}, G_\delta)^t(L_0, K_0) := (\mathcal{L}, G_\delta)^t \circ \cdots \circ (\mathcal{L}, G_\delta)(L_0, K_0).
\]

Thus, an orbit \( \gamma(L_0, K_0) \) of the time-one map \((\mathcal{L}, G_\delta)\) is given by

\[
\gamma(L_0, K_0) = \{(\mathcal{L}, G_\delta)^t(L_0, K_0)\}_{t \in \mathbb{N}},
\]

where the sequence \( \{(L_t, K_t)\}_{t \geq 0} \) is also called a growth path.

The following properties are usually not stated explicitly, since they are simple and immediate consequences of the underlying properties. The assumption of full employment in each period implies that the change of employment is equal to the change of the population. Hence, employment in each period grows independently of the capital stock and of income. Since \( F \) is homogeneous of degree one, the map \((\mathcal{L}, G_\delta)\) is homogeneous of degree one and the origin \((L, K) = (0, 0)\) is a fixed point.

If \( n \neq 0 \), then the origin is the only fixed point. If \((\bar{L}, \bar{K})\) is a fixed point for \( n = 0 \), then \((\lambda \bar{L}, \lambda \bar{K})\) is also a fixed point for any \( \lambda > 0 \). Such fixed points do not describe interesting long run situations of economic growth. However, since the system \((\mathcal{L}, G_\delta)\) is homogeneous of degree one, the notion of balanced growth for the long run development of a growing economy replaces the concept of stationarity, being defined by orbits along which capital and labor grow at the same constant rate.
Define the growth rate of capital by the usual formula

\[ n_K := (K_{t+1} - K_t)/K_t. \]

The capital accumulation law (2.2) implies

\[ 1 + n_K = \frac{G_s(L_t, K_t)}{K_t} = (1 - \delta) + \bar{s}F(L_t, K_t). \]

Rewriting \( G_s(L, K) \) using the homogeneity property, one has

\[ G_s(L, K) = K[1 + n_K(K)] \]

so that the growth rate of capital becomes

\[ n_K(K) := \frac{\bar{s}f(k)}{k}, \]

where \( k = K/L \) is the capital-labor ratio or the so called capital intensity. Thus, the growth rate of capital at any time is homogeneous of degree zero in \((L, K)\) and determined completely by the capital-labor ratio \( K/L \).

The requirement that the capital stock grows at the same rate as the population, i.e.

\[ \frac{K_{t+1} - K_t}{K_t} = (1 + n_K(K)) \frac{K_t}{L_t} = n = \frac{L_{t+1} - L_t}{L_t} \]

implies that the capital labor ratio is constant over time. This follows from

\[ k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{(1 + n_K(K_t))K_t}{(1 + n)L_t} = k_t, \quad \text{for all } t. \]

**Definition 2.1** A growth path \( \gamma(L_0, K_0) \) of (2.2) with \((L_0, K_0) \geq 0\) is called balanced, if

\[ \frac{K_{t+1} - K_t}{K_t} = n_K \left( \frac{K_t}{L_t} \right) = n = \frac{L_{t+1} - L_t}{L_t} \]

for all points \((L_t, K_t)\), \( t \in \mathbb{N} \) of the orbit \( \gamma(L_0, K_0) \).

Let \( \bar{k} > 0 \) be the capital labor ratio associated with a balanced growth path \( \gamma(L_0, K_0) \) of the system (2.2). Then, according to Definition 2.1, the initial capital stock satisfies \( K_0 = \bar{k}L_0 \). Moreover, capital and labor evolve over time with a constant capital labor ratio, that is, for all \( t \) one has \( K_t = \bar{k}L_t \). In other words, in state space \( \mathbb{R}^2_+ \), capital and labor evolve along the ray \( K = \bar{k}L \), as illustrated in Fig. 1 for \( n > 0 \).

### 2.2 Existence of balanced growth paths

Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \), \( f(\frac{K}{L}) := F(1, \frac{K}{L}) \) denote the so called intensive form\(^1\) of the production function \( F \) and consider an economy of the above type with population growth rate \( n > -1 \) and depreciation rate \( 0 \leq \delta \leq 1 \).

**Lemma 2.1** A growth path \( \gamma(\bar{L}, \bar{K}) \) is balanced, if and only if the capital labor ratio \( \bar{k} = \bar{K}/\bar{L} \) is strictly positive and satisfies \( \bar{s}f(\bar{k}) = (n + \delta)\bar{k} \).

\(^1\)As is customary in this literature, the capital labor ratio \( \frac{K}{L} \equiv k \) is also called the capital intensity, where both terms are used interchangeably and equivalently.
Figure 1: Balanced growth path.

Proof:
By Definition 2.1, \( (L, K) \) is a balanced growth path, if the growth rate of capital is constant over time, that is

\[
K_t = K_0 (1 + n)^t \quad \text{for all } (L_t, K_t) \in \gamma(L, K)
\]

implying that \( k = k_0 = \frac{K}{L} \) satisfies \( s f(k) = (n + \delta)k \). On the other hand \( k_0 = \bar{k} \) and \( k_t = \bar{k} \Rightarrow k_{t+1} = \bar{k} \) implies by induction \( k_t = \bar{k} \) for all \( t \) and hence \( n_K \left( \frac{K_t}{L_t} \right) = n_K(\bar{k}) = n \) for all \( t \in \mathbb{N} \). QED.

Lemma 2.1 states that balanced growth paths exist, if and only if there exists a positive capital intensity \( \bar{k} > 0 \) such that the average capital productivity satisfies \( s f(k) = (n + \delta)/s \). If \( \nu(k) := \frac{k}{f(k)} \) denotes the so called capital coefficient, the growth rate of the capital stock \( n_K(k) \) can be rewritten as \( n_K(k) = \frac{s}{\nu(k)} = \frac{n + \delta}{s} \). Lemma 2.1 then implies that balanced growth paths must satisfy \( \frac{s}{\nu(k)} = n + \delta \) or, equivalently,

\[
\text{savings propensity} = \frac{\text{capital coefficient}}{\text{rate of population growth} + \text{rate of depreciation}}.
\]

Since \( (n, \delta, s) \) are given parameters of the economy, these considerations imply that the capital productivity \( f(k)/k \) (or the capital out ratio \( \nu(k) \)) must be sufficiently flexible as a function of \( k \) to guarantee existence of steady states for all economies. To obtain a general existence result the following generalization of the so called Inada\(^2\) conditions becomes useful.

Definition 2.2 A continuous function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is said to satisfy the weak Inada conditions, if

\[
(a) \lim_{k \to 0} \frac{f(k)}{k} = \infty \quad \text{and} \quad (b) \lim_{k \to \infty} \frac{f(k)}{k} = 0.
\]

\(^2\)A differentiable function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is said to satisfy the Inada conditions, if \( (a) \lim_{k \to 0} f'(k) = \infty \) and \( (b) \lim_{k \to \infty} f'(k) = 0 \). They were introduced in Inada (1963) for the first time. Clearly, the Inada conditions imply the weak Inada conditions for any differentiable function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \).
The weak Inada conditions have a straightforward economic interpretation. Since \( k/f(k) \) measures the capital input requirement for a unit of output, they imply that under efficient substitution between labor and capital, the capital productivity/the capital coefficient may attain any positive value (assuming continuity of \( f \)). In economic terms this means that capital can be substituted by labor without restrictions. Thus, to produce a unit of output efficiently any positive level of capital \( K > 0 \) requires a positive input level \( L > 0 \) of labor. In other words, labor is an essential input factor implying \( F(0, K) = 0 \) for all \( K \geq 0 \) (see Panel (a) of Figure 2), while capital does not have to be an essential input factor.

Figure 2: Isoquants and weak Inada conditions

To describe the implication for the unit isoquant geometrically, suppose that both limits of \( f(k)/k \) are finite and positive, i.e.

\[
(a) \quad \lim_{k \to 0} \frac{f(k)}{k} = M < \infty \quad \text{and} \quad (b) \quad \lim_{k \to 1} \frac{f(k)}{k} = M > 0.
\]

Then the capital input requirement function \( \kappa(k) := k/f(k) \) has the limiting properties

\[
(a) \quad \lim_{k \to 0} \kappa(k) = \lim_{k \to 0} \frac{k}{f(k)} = \frac{1}{M} \quad \text{and} \quad (b) \quad \lim_{k \to \infty} \kappa(k) = \lim_{k \to \infty} \frac{k}{f(k)} = \frac{1}{M},
\]

with \( 0 < 1/M < 1/M < \infty \), implying a capital input requirement function \( \kappa : \mathbb{R}_+ \to \mathbb{R}_+ \) which is not surjective. In this case, the associated unit isoquant has the form as drawn in panel (b) of Figure 2, starting at the point \( (0, 1/M) \) becoming tangent or equal to the horizontal line at \( 1/M \).

The following Proposition 2.1 provides the weakest existence result.

**Proposition 2.1** Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be continuous, and the map \( k \mapsto f(k)/k \) be decreasing. The equation \((n + \delta)k = s f(k)\) has a unique positive solution \( \bar{k}(n, \delta, s) > 0 \) if and only if

\[
0 \leq \lim_{k \to \infty} \frac{f(k)}{k} < \frac{n + \delta}{s} < \lim_{k \to 0} \frac{f(k)}{k} \leq \infty.
\]
Proof: Existence follows from (2.3) and the intermediate value theorem. Uniqueness is consequence of the monotonicity of $k \mapsto f(k)/k$. QED.

Obviously, the weak Inada conditions imply (2.3). On the other hand, for all $(n, \delta, s)$ with $n + \delta > 0$ and $0 < s \leq 1$ the range $\frac{n + \delta}{s}$ is $\mathbb{R}_+$, such that the Inada conditions are necessary to get steady states for all parameters $(n, \delta, s)$ with $n + \delta > 0$ and $0 < s \leq 1$.

**Proposition 2.2** Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, and the map $k \mapsto f(k)/k$ be decreasing. Then the following statements are equivalent:

(i) $f$ satisfies the weak Inada conditions,

(ii) for every $(n, \delta, s)$ with $n + \delta > 0$ and $0 < s \leq 1$, there exists a unique $\bar{k} = \bar{k}(n, \delta, s) > 0$ with $s f(k) = (n + \delta)k$.

Proof: If $f$ satisfies the weak Inada conditions, then the map $k \mapsto f(k)/k$ is surjective. Since this map is decreasing by assumption, it is also injective. This implies (ii). On the other hand, if $f$ satisfies (ii), the map $k \mapsto f(k)/k$ has to be one-to-one on $\mathbb{R}_+$. Since it is decreasing, it follows that $f$ must satisfy the weak Inada conditions. QED.

Observe that any differentiable and strictly concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly monotonically increasing with $f'(k) < f(k)/k$ for all $k$. Hence

$$\left( \frac{f(k)}{k} \right)' = \frac{f'(k)k - f(k)}{k^2} < 0 \quad \text{for all } k > 0,$$

implying that the capital product $f(k)/k$ is strictly monotonically decreasing in $k$. It then follows from Proposition 2.2 that balanced growth paths $\gamma(L, K)$ with $\bar{k} = K/L$ exist for all possible choices of $n + \delta > 0$ and $\bar{s}$.

If either Conditions (a) or (b) of the weak Inada conditions is not satisfied, then the existence of $\bar{k}$ may fail, see Fig. 3. However, as long as the weak Inada condition $\lim_{k \to 0} \frac{f(k)}{k} = \infty$ holds, the existence of $\bar{k} > 0$ can be guaranteed for sufficiently small $\bar{s}$.

### 2.3 Stability of steady states in intensity form

In order to analyze the stability of balanced growth paths, let us first examine the dynamics of the corresponding capital intensities (the capital labor ratios) $k = K/L$. Throughout this section we assume the production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ to be differentiable and concave (and therefore non-decreasing).

**Definition 2.3** The dynamical system in intensity form associated with $(L, G_s)$ is defined by the time-one map

$$G_s : \mathbb{R}_+ \to \mathbb{R}_+, \quad k \mapsto G_s(k) = \frac{1}{1 + \frac{1}{s}}[(1 - \delta)k + \bar{s}f(k)].$$
Since \( f(k) \) and \((1-\delta)k\) are both concave and non-decreasing functions, the time-one map \( G_s \) again is concave and non-decreasing. Moreover, \( G_s(0) = \frac{s}{1+n} f(0) \) and 
\[
G_s'(k) = \frac{1}{1+n} [1 - \delta + s f'(k)] \geq 0 \quad \text{for all } k \in \mathbb{R}_+.
\]

For \( \delta < 1 \), \( G_s \) is strictly monotonically increasing. Observe that \( \bar{k} = G_s(\bar{k}) \) is a steady state of \( G_s \), if and only if \((n+\delta)\bar{k} = \bar{s}f(\bar{k})\). It follows from Lemma 2.1, that balanced growth paths of the system \((L, G_s)\) correspond exactly to steady states of the associated system \( G_s \). The next lemma follows from Proposition 2.2 and our assumptions on the production function \( f \).

**Lemma 2.2** Let \( n+\delta > 0 \) and \( 0 < \bar{s} \leq 1 \) be arbitrary. Then \( G_s \) has a unique positive steady state \( \bar{k} = \bar{k}(n, \delta, \bar{s}) > 0 \), if and only if the concave production function \( f \) satisfies the weak Inada conditions.

Fig. 4 indicates geometrically that the dynamics of the capital intensities always converges monotonically to the unique positive steady state \( \bar{k} \). The following proposition makes this graphical observation precise.

**Proposition 2.3** Let \( n+\delta > 0 \) and \( 0 < \bar{s} \leq 1 \) be arbitrary and assume, in addition that the concave production function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) satisfies the weak Inada conditions. Then \( G_s \) has a unique positive steady state \( \bar{k} = \bar{k}(n, \delta, \bar{s}) > 0 \) which is globally asymptotically stable on \( \mathbb{R}_{++} \). Moreover, all orbits \( \gamma(k_0), k_0 \in \mathbb{R}_{++} \) are monotonic sequences.

**Proof:**
The existence of a unique positive steady state \( \bar{k} > 0 \) follows from Lemma 2.2. Let \( k > \bar{k} \) be arbitrary. Since \( G_s \) is non-decreasing and concave, \( \bar{k} \leq G_s(k) < k \) and \( G_s^2(k) \leq G_s(k) \). It follows by induction that \( \bar{k} \leq G_s^t(k) \leq G_s^{t-1}(k) \) for all \( t \in \mathbb{N} \). Hence, the sequence \( \{G_s^t(k)\}_{t \in \mathbb{N}} \) is monotonically decreasing. Since it is bounded from below by \( \bar{k} \), there exists a unique limit point \( k_0 \) with \( \lim_{t \to \infty} G_s^t(k) = k_0 \). The limit point must be a steady state of \( G_s \). Since \( \bar{k} \) was the unique steady state, \( \bar{k} = k_0 \). For \( k < \bar{k} \) the argument is analogous.

QED.
It was seen in the previous section that there are two reasons why existence of a positive steady state \( k > 0 \) of \( G_s \) may fail. These are violations of either part of the weak Inada conditions. First, let \( f(0) = 0 \) and suppose that \( \lim_{k \to 0} f(k)/k \leq M < \infty \), such that condition (a) of the weak Inada conditions is violated. Since \( f \) is concave, \( f'(k)k \leq f(k) \) for all \( k \) yields (taking right limits) \( f'(0) \leq M \). This implies

\[
G'_s(0) = \frac{1}{1 + n} [1 - \delta + \bar{s} f'(0)] \leq 1 \iff f'(0) \leq \frac{n + \delta}{\bar{s}}.
\]

Since \( G_s \) is concave with \( G_s(0) = 0 \), 0 is the only steady state if \((n + \delta)/\bar{s} \geq M\). Consequently, if the savings propensity \( s \) is too small, 0 is globally asymptotically stable, meaning that the economy will impoverish in the long run (see Fig. 4 (b)). Second, suppose that \( f(0) \geq 0 \) and \( \lim_{k \to \infty} f(k)/k \geq n + \delta > 0 \) such that \( f \) violates condition (b) of the weak Inada conditions. Then there exists \( \bar{s} > 0 \) with \( f(k)/k > \frac{n + \delta}{\bar{s}} \) for all \( k > 0 \). The monotonicity and the concavity of \( G_s \) imply that \( G_s(k) > k \) for all \( k > 0 \), see Fig. 4. Thus, the capital intensities \( k_t = G_s'(k_0) \) diverge to infinity for all initial values \( k_0 > 0 \) as \( t \) tends to infinity. Therefore, output per capita \( y_t = f(k_t) \) as well as consumption per capita \( c_s(k_t) = (1 - s)f(k_t) \) tend to infinity as well. Hence, no positive steady state \( k \) exists where all variables grow at the same constant rate (see Fig. 4 (c)).

### 3 Stable Balanced Growth

Given the results on existence and uniqueness of balanced growth paths, it was a tempting conclusion that the stability of the balanced growth path would follow also from the stability in intensive form. Unfortunately, this is not the case. As the results of the next section show, the stability of balanced growth requires additional conditions which cannot be captured by the intensity conditions alone. The instability arises in a structural way in all models in intensity form as a consequence of the reduction of the dimension.

Let us return to the dynamical system (2.2) which describes the evolution of the capital stock \( K \) and of the population \( L \), and consider a more general continuous aggregate savings function \( S(K, L) \) which is homogeneous of degree one. Define \( s(k) := S(k, 1) \) to be the associated per capita
Theorem 3.1 de\textsuperscript{\textregistered}ned by

The next Theorem links the stability of balanced growth paths to the elasticity of the production function, where \( k = \frac{K}{L} \) as before\textsuperscript{3}. Then, the dynamics in state space is described by the difference equations

\[
\begin{align*}
K_{t+1} &= (1 - \delta)K_t + s \left( \frac{K_t}{L_t} \right) L_t, \\
L_{t+1} &= (1 + n)L_t.
\end{align*}
\]  

An orbit \( \gamma(L_0, K_0) \) is said to converge to the balanced growth path defined by \( \bar{k} \), if the difference

\[
\Delta_t := K_t - \bar{k}L_t = G_s(L_{t-1}, K_{t-1}) - \bar{k}L_t
\]

between the actual capital stock \( K_t = G_s(L_{t-1}, K_{t-1}) = (1 - \delta)K_{t-1} + S(K_{t-1}, L_{t-1}) \) and the balanced capital stock \( \bar{k}L_t \) converges to zero for \( t \to \infty \).

Let \( \bar{k} > 0 \) denote an asymptotically stable fixed point of the dynamical system \( G_s \) associated with \( G_s, i. e. of \ G_s(k) = \frac{1}{1 + n}((1 - \delta)k + s(k)) \).

Using the definition of \( G_s \), one has

\[ \Delta_t = L_t[G_s(k_{t-1}) - \bar{k}] \quad \text{for all } t. \]

If an orbit of \( G_s \) is monotonically increasing, the differences \( \Delta_t \) are either positive for all \( t \) or negative for all \( t \), depending on whether the initial \( \Delta_0 \) is positive or negative. As seen above, the special case \( \Delta_0 = K_0 - \bar{k}L_0 = 0 \) corresponds to balanced growth paths of (2.2), implying that \( \Delta_t = 0 \) for all times \( t \), see Fig. 1.

The next Theorem links the stability of balanced growth paths to the elasticity of the production function \( f \) at the steady state \( \bar{k} \) of \( G_s \). Recall that the elasticity of a function \( f \) at some point \( k \) is defined by \( E_f(k) = f'(k)k/f(k) \).

**Theorem 3.1** Let \( s \) be differentiable and let \( \bar{k} \) be an asymptotically stable fixed point of \( G_s \). Let\textsuperscript{4} \( B(\bar{k}) \cap (G_s)^{-1}(\{\bar{k}\}) = \{\bar{k}\} \), where \( B(\bar{k}) \) is the basin of attraction of \( \bar{k} \) and \( (G_s)^{-1}(\{\bar{k}\}) \) is the preimage of \( \bar{k} \). Consider the time-one map \( (L, G_s) \) as given in (2.2). Let \( \gamma(L_0, K_0) \) be an arbitrary orbit of \( (L, G_s) \) with \( K_0/L_0 \in B(\bar{k}), K_0/L_0 \neq \bar{k} \), implying \( \Delta_0 = K_0 - \bar{k}L_0 \neq 0 \). Then the following holds:

If \( E_s(\bar{k}) < \frac{\delta}{n + \delta} \), then \( \lim_{t \to \infty} \Delta_t = 0; \)  

If \( E_s(\bar{k}) > \frac{\delta}{n + \delta} \), then \( \lim_{t \to \infty} |\Delta_t| = \infty, \)

where \( E_s(\bar{k}) = \frac{\bar{k}s'(\bar{k})}{s(\bar{k})} \) denotes the elasticity of the function \( s \) at the steady state \( \bar{k} \).

**Proof:**

Let \( k_0 > 0 \) and \( k_0 \neq \bar{k} \) arbitrary but fixed. Under the hypotheses of this Theorem, one has

\[ \Delta_{t+1} = K_{t+1} - \bar{k}L_{t+1} = L_{t+1}[G_s(k_t) - \bar{k}], \quad t \in \mathbb{N}, \]

where \( \bar{k} \) denotes the steady state of \( G_s \). Then,

\[
\frac{\Delta_{t+1}}{\Delta_t} = (1 + n) \frac{G_s(k_t) - \bar{k}}{k_t - \bar{k}}.
\]

\textsuperscript{3}We continue to use the notation \( G_s \) and \( G_s \) for the case where \( s \) is now a per capita savings function \( s(k) \). No confusion with the Solow case where \( s(k) = sf(k) \) should arise.

\textsuperscript{4}This assumption is always satisfied, if \( s \) is strictly increasing.
$k_t$ converges to $\bar{k}$ since $k_0 \in B(\bar{k})$. Therefore,
\[
\lim_{t \to \infty} \frac{\Delta t + 1}{\Delta t} = (1 + n)G_s'(\bar{k}).
\]
This implies that \(\frac{\Delta t + 1}{\Delta t} - (1 + n)G_s'(\bar{k})| < \epsilon\) for all $t$ larger than some $t_0$. It follows that
\[
[(1 + n)G_s'(\bar{k}) - \epsilon]|\Delta t| < |\Delta t + 1| < [(1 + n)G_s'(\bar{k}) + \epsilon]|\Delta t|, \quad t \geq t_0,
\]
and by induction
\[
[(1 + n)G_s'(\bar{k}) - \epsilon]|\Delta t + t_0| < |\Delta t + 1| < [(1 + n)G_s'(\bar{k}) + \epsilon]|\Delta t|, \quad \tau > 0.
\]
Now, if $(1 + n)G_s'(\bar{k}) < 1$, then $(1 + n)G_s'(\bar{k}) + \epsilon < 1$ for sufficiently small $\epsilon$ such that $\lim_{t \to \infty} \Delta t = 0$. On the other hand, if $(1 + n)G_s'(\bar{k}) > 1$, then $(1 + n)G_s'(\bar{k}) - \epsilon > 1$ for sufficiently small $\epsilon$ and hence $\lim_{t \to \infty} |\Delta t| = \infty$. The rest of the statement follows from
\[
(1 + n)G_s'(\bar{k}) < 1 \iff E_s(\bar{k}) < \frac{\delta}{n + \delta},
\]
and the fact that $k_0 \in B(\bar{k})$, $k_0 \neq \bar{k}$ was arbitrary. QED.

Theorem 3.1 applies to functions $G_s$ which are strictly increasing or strictly decreasing in a neighborhood of a locally asymptotically stable fixed point. As an immediate implication of Theorem 3.1 one obtains the following corollary for the Solow model.

**Corollary 3.1** A balanced growth path of the Solow model is stable if
\[
E_f(\bar{k}) < \frac{\delta}{n + \delta},
\]
and unstable if
\[
E_f(\bar{k}) > \frac{\delta}{n + \delta}.
\]

For the Solow model, the elasticity of the savings function always coincides with the elasticity of the production function. Since the production function $f$ is strictly concave, its elasticity $E_f(k)$ is always less than one. Therefore, for the Solow model, balanced growth paths are always stable for non negative growth rates $n \leq 0^5$. Notice, however, that for positive rates $n$ the stability of the balanced growth path is not always guaranteed and that the distance from the balanced path diverges to infinity in case of (3.4) in spite of the fact that the intensity converges.

In order to understand the implications of the above result for the unstable case, consider two economies (with a balanced path $\hat{k} > 0$) which are identical in all of their characteristics including the initial level of labor supply $L_0$, but which differ only in their initial capital with $K_0^1/L_0 < \hat{k} < K_0^2/L_0$. Then, in the long run the capital stock $\hat{K}_t$ of both economies will be infinitely far away (below resp. above) from the balanced path implying that their difference will also become infinite. In any period $t$ the rate of capital accumulation of the economy with the lower initial capital will always be larger than $1 + n$ while the rate of capital accumulation of the economy with higher initial capital will always be less than $1 + n$. However, the level of capital of the economy with initial undercapitalization will never catch up with that of an economy with initial overcapitalization. In other

---

5This was recognized already by Deardorff (1970); for positive population growth see also Böhm & Wenzelburger (1999).
words, the *capitalization* of two identical economies will be diverging. Therefore, the property of convergence of capital intensities alone is a weak concept which does not guarantee the convergence to the balanced path. This feature is a common property of all one dimensional models in intensity form derived from two factor models of growth which has largely been ignored or overlooked in the literature.\(^6\)

While the proof of Theorem 3.1 is somewhat technical and not too revealing, the following Lemma provides a useful check of the local asymptotic stability of balanced growth paths for arbitrary models in intensity form. It also exhibits the role of the elasticity of the mapping \(G\) and of the rate of population growth for the stability of the balanced path.

**Lemma 3.1** Consider a twice differentiable time one map \(G: \mathbb{R}_+ \rightarrow \mathbb{R}_+\) of a growth model in intensity form with a positive fixed point \(\bar{k} = G(\bar{k})\) where the labor force grows at the rate \(n > -1\). Then, the positive balanced growth path associated with \(\bar{k}\) is asymptotically stable if

\[
\max(|(1 + n)G'(\bar{k})|, |G'(\bar{k})|) < 1 \tag{3.7}
\]

**Proof:**

Define an associated two dimensional system

\[
k_{t+1} = G(k_t) \tag{3.8}
\]

\[
\Delta_{t+1} = (1 + n) \frac{G(k_t) - \bar{k}}{k_t - \bar{k}} \Delta_t, \tag{3.9}
\]

where \(\Delta := (k - \bar{k})L\) measures the deviation from the ray \(\bar{k}L\) associated with the balanced path. Then:

i) A positive fixed point \(\bar{k}\) of \(G_s\) defines a positive balanced path if and only \((\bar{k}, 0)\) is a fixed point of \((3.8), (3.9)\)

ii) \((3.8) - (3.9)\) has the two eigenvalues \(\lambda_1 = G'(\bar{k})\) and \(\lambda_2 = (1 + n)G'(\bar{k})\).

Therefore, for any \(L_0 > 0\) and sufficiently small \(\epsilon\) such that

\[
|k_0 - \bar{k}| = \frac{|\Delta_0|}{L_0} \leq \frac{\epsilon}{L_0}
\]

one has \(\lim_{t \to \infty} \Delta_t = 0\). QED.

Observe that Lemma 3.1 applies to very general one dimensional growth models in intensity form, including OLG models and optimal growth models. In addition, it provides clear cut information about the asymptotic stability of any number of hyperbolic fixed points of the augmented system \((3.8) - (3.9)\). In other words, if the elasticity of \(G\) at any steady state is greater than \(1/(1+n)\), then the basin of attraction of the balanced path is empty. Thus, even in the case of multiple balanced growth paths the stability of each one depends on the elasticity of \(G\) at each fixed point.\(^7\) It may very well be the case that none of them is stable!

\(^6\)Deardorff (1970) and Jensen (1994) are two notable exceptions.

\(^7\)Compare Galor (1996).
Stable growth paths in the Solow model with different technologies

**Example 3.1** Consider linear production functions of the form

\[ f(k) = a + bk, \quad a, b > 0. \]

Since

\[ \lim_{k \to 0} \frac{f(k)}{k} = \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{f(k)}{k} = b, \]

\( f \) does not satisfy property (b) of the weak Inada conditions. However, for the associated time one map \( G_s \)

\[ k_{t+1} = \frac{1}{1 + n} \left( (1 - \delta)k_t + s(a + bk_t) \right) \]

one finds a unique steady state

\[ \bar{k} = \frac{sa}{n + \delta - bs} \]

which is greater than zero for \( n + \delta > sb \). Under this condition a unique positive balanced growth path exists. The elasticity of \( f \) on the balanced path is \( E_f(\bar{k}) = \frac{sb}{n + \delta} \). Therefore, the balanced growth paths are

stable if \( sb < \delta \) and unstable if \( sb > \delta \).

Thus, stable permanent balanced growth occurs for all parameter values whenever

\[ n + \delta > sb \quad \text{and} \quad sb < \delta. \]

The balanced path exists but is unstable for

\[ n + \delta > sb > \delta \]

with capital diverging from the balanced path for any \( K_0/L_0 \neq sa/(n + \delta - bs) \) since

\[ \Delta_{t+1} := K_{t+1} - \bar{k}L_{t+1} \]

\[ = \Delta_t + \bar{k}(n + \delta - sb)L_t + (sb - \delta)\bar{k}nL_t \]

\[ = \Delta_t + (sb - \delta)(\bar{k} - \bar{k}L_t) = (1 + sb - \delta)\Delta_t. \]

**Example 3.2** Consider isoelastic production functions of the form

\[ f(k) = Ak^\alpha, \quad A > 0, \quad 0 < \alpha < 1. \]

Then \( \frac{f(k)}{k} = Ak^{\alpha-1} \) and

\[ \lim_{k \to 0} \frac{f(k)}{k} = \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{f(k)}{k} = 0. \]

Hence, isoelastic production functions satisfy the weak Inada conditions. In this case the unique positive steady state of the associated system \( G_s \) is given by

\[ \bar{k} = \left( A \frac{s}{n + \delta} \right)^{\frac{1}{1 - \alpha}}, \quad n + \delta > 0. \]

\[ \text{Compare for example Barro & Sala-I-Martin (1995) and Aghion & Howitt (1998)} \]
Since $E_f(k) \equiv \alpha$ for all $k$, balanced paths are stable for all $0 < s \leq 1$, if

$$\alpha < \frac{\delta}{n + \delta}.$$ 

Therefore, they are stable for sufficiently small growth rates $n$ of the population, in particular for all $n \leq 0$. However, for $n > 0$,

$$\alpha > \frac{\delta}{n + \delta}$$

implies an unstable balanced growth path for any $s$, with capital diverging whenever

$$\frac{K_0}{L_0} \neq \bar{k} = \left( \frac{A - s}{n + \delta} \right)^{\frac{1}{1-n}}.$$

Thus:

$$\Delta_{t+1} := K_{t+1} - \bar{k}L_{t+1}$$

and:

$$\Delta_t + (s(f(k_t) - f(\bar{k}))(n + \delta)\bar{k}L_t + \delta k_t - n\bar{k})L_t$$

are:

$$\Delta_t + \left( s \frac{f(k_t) - f(\bar{k})}{k_t - \bar{k}} - \delta \right) (k_t - \bar{k})L_t$$

$$\Delta_t \left( s \frac{f(k_t) - f(\bar{k})}{k_t - \bar{k}} - \delta \right)$$

Since

$$f'(\bar{k}) = \alpha \frac{n + \delta}{s},$$

one has

$$\lim_{t \to \infty} s \frac{f(k_t) - f(\bar{k})}{k_t - \bar{k}} - \delta = \alpha(n + \delta) - \delta > 1.$$

Therefore, for all $t \geq t_0$

$$|\Delta_{t+1}| > |\Delta_t|.$$ 

Figure 5 summarizes the qualitative results of Examples 3.1 and 3.2, by displaying the ranges of parameters (shaded region) for which unstable positive balanced growth paths exist.

**Example 3.3** Consider the case of Leontief production functions of the form

$$f(k) = \min \{a, bk\}, \quad a > 0, \quad b > 0,$$

with

$$\frac{f(k)}{k} = \min \left\{ \frac{a}{k}, b \right\}.$$ 

As can be seen from Fig. 6, the weak Inada conditions are violated. Therefore a fixed point $\bar{k}$ exists, if and only if

$$0 < \frac{n + \delta}{s} \leq b.$$ 

Since the savings propensity $s$ is bounded by 1, let us assume that $(n + \delta)/b \leq 1$. Otherwise no savings propensity can lead to a positive steady state.

The evolution of the capital intensity is driven by

$$G_s(k) = \begin{cases} 
\frac{1-\delta+sbk}{1+n}, & k \leq \frac{a}{b}, \\
\frac{1-\delta}{1+n}k + \frac{sa}{1+n}, & k > \frac{a}{b}.
\end{cases} \quad (3.10)$$

There are three distinct cases (a) – (c) depending on the savings propensity $s$ (see Figure 7).
Case (a): $s > \frac{n+\delta}{b}$.

Then, $1 - \frac{\delta + sb}{1+n} > 1$ and there exists a unique fixed point $\bar{k}(n, \delta, s) > \frac{n}{b}$ of $G_s$, see Fig. 7(a). The case when $k$ is strictly larger than $\frac{n}{b}$ is usually referred to as a situation of over-accumulation, where labor is the binding input factor.

Case (b): $s = \frac{n+\delta}{b}$.

Then, $1 - \frac{\delta + sb}{1+n} = 1$ and $G_s$ has a continuum of fixed points, namely for each $\bar{k} \in [0, \frac{n}{b}]$, see Fig. 7(b). Notice that $\frac{n}{b}$ is a particular fixed point of $G_s$. All steady states with $0 < k < \frac{n}{b}$ have excess usage (input) of labor, while capital is fully employed representing the binding input factor. Such situations are often referred to as cases of under-accumulation with unemployment (see Figure 6).

Case (c): $s < \frac{n+\delta}{b}$.

The slope of the first function $\frac{1-\delta + sb}{1+n}$ in (3.10) is less than 1, so that the graph $G_s$ lies below
As is known from production theory, Leontief production functions do not allow efficient substitution between labor and capital, the capital intensity \( \frac{K}{L} \) being the unique efficient factor input ratio. The associated isoquants of the Leontief production function

\[ F(L, K) = Lf(k) = \min\{aL, bK\}, \quad (L, K) \in \mathbb{R}^2_+ \]

are straight lines forming right angles along the ray \( \frac{a}{b}L \). Therefore (see Fig. 8), any point \((L, K)\) with a capital-labor ratio \( k = \frac{K}{L} \) different from \( \frac{a}{b} \) is inefficient, i.e. the same aggregate output could be reached with either less capital stock or less labor input. Thus growth paths lying above the ray \( \frac{a}{b}L \) are paths with over-accumulation of capital, whereas paths lying below are paths with under-accumulation (relative to efficient factor inputs). In order to examine the stability of balanced growth paths, we consider the unique steady state and the continuum of steady states.
growth paths in the Leontief case, the elasticity of the production function is given by

\[ E_f(k) = \begin{cases} 
1, & k < \frac{a}{b} \\
0, & k > \frac{a}{b} 
\end{cases} \]

It follows from Theorem 3.1 that, for growing populations \( n > 0 \), the balanced growth paths with over-accumulation \( \delta(n, \delta, s) > \frac{a}{b} \) are always stable in state space. If balanced growth paths with under-accumulation \( \delta(n, \delta, s) < \frac{a}{b} \) exist, there must be a continuum of them corresponding to case (b). However, any two of them are mutually diverging along different rays. Therefore, the Harrod–Domar growth model has a stable balanced growth path with over-accumulation whenever \( \frac{n + \delta}{s} < b \). In all other cases stable balanced growth does not exist.

4 Stable Balanced Growth with random perturbations

In this section we show that the issues of the stability of intensity form models and of their respective state space versions arise in the same way for stochastic growth models. In other words, convergence (in a stochastic sense) for models in intensity form does not guarantee convergence in state space. Moreover the structural source of the instability can be identified in the same way arising from a property of the per capita savings function. This translates into a condition of the elasticity of the production function for the Solow model\(^9\).

Consider the typical random growth model (in absolute terms) as described by two random difference equation in the space of capital \( K \) and labor \( L \) given by

\[ \begin{align*}
K_{t+1} &= (1 - \delta)K_t + s \left( \xi_t, \frac{K_t}{L_t} \right) L_t, & K_0 \text{ given}, \\
L_{t+1} &= (1 + n_t)L_t, & L_0 \text{ given},
\end{align*} \]

with an exogenous noise process \( \omega := \{(\xi_i, n_i)\}_{i \in \mathbb{Z}} \),

- a per capita savings function \( s : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ : (\xi, k) \mapsto s(\xi, k), \) where \( k \geq 0 \) is the capital intensity and \( \xi \in [\xi_{\min}, \xi_{\max}] \subset (0, \infty) \) is a parameter, describing random productivity, random income, or random savings behavior,
- a random rate of population growth, \( n \in [n_{\min}, n_{\max}] \subset (-1, \infty) \),
- and a rate of depreciation\(^{10} \) \( 0 \leq \delta \leq 1 \).

A path of random parameters is assumed to follow a stationary stochastic process described by \( \omega := \{\omega_t\}_{t \in \mathbb{Z}} = \{(\xi_t, n_t)\}_{t \in \mathbb{Z}} \in \Omega := \mathbb{Z}^2, \mathbb{Z} := [\xi_{\min}, \xi_{\max}] \times [n_{\min}, n_{\max}], \) such that \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a probability space of random perturbations. By the assumption of an exogenous random growth rate \( n_t, \ t \in \mathbb{Z}, \) the labor force follows an exogenous time path \( L_t = L(t, \omega, L_0) \) driven by the linear random difference equation \( L_{t+1} = (1 + n_t)L_t. \) A solution of (4.1), (4.2) is denoted by \( (K(t, \omega, K_0, L_0), L(t, \omega, L_0)) \).

\(^9\)This section uses the description, techniques, and results of Schenk-Hoppé & Schmalfuss (2001), Böhm & Wenzelburger (2002)

\(^{10}\)To reduce notation we chose a deterministic rate of capital depreciation since a random one does not imply additional results.
The induced random growth model in intensity\textsuperscript{11} form is given by

\[ k_{t+1} = h(\xi_t, n_t; k_t) := \frac{(1 - \delta)k_t + s(\xi_t, k_t)}{1 + n_t} \quad \text{with } k_0 := \frac{K_0}{L_0} \]  

(4.3)

and a solution of (4.3) is denoted by \( k(t, \omega, k_0) \). Using a measurable invertible shift–operator \( \theta : \Omega \to \Omega, \theta \omega = \{ (\xi_{t+1}, n_{t+1}) \}_{t \in \mathbb{Z}} \), the parameters in period \( t \) are \( (\xi_t, n_t) = \omega_t = (\theta^t \omega)_0 \). With this notation (4.3) takes the form \( k_{t+1} = h(\omega_t, k_t) = h((\theta^t \omega)_0, k_t) \).

In a stochastic environment the analogue to a steady state in a deterministic system is either a random fixed point, where time paths are invariant against the shift operator, or a stationary solution, where the distribution is time invariant. Following (Schmalfuß 1996, 1998), a random fixed point of \( h \) is a random variable \( k_* : \Omega \to \mathbb{R}_+ \) on the probability space of the noise process \( (\Omega, \mathcal{F}, \mathbb{P}) \) such that \( \mathbb{P} \)-almost surely\textsuperscript{12}

\[ k_*(\theta \omega) = h(\omega_0, k_*(\omega)) \quad \text{for all } \omega \in \Omega', \]  

(4.4)

where \( \Omega' \subset \Omega \) is a \( \theta \)-invariant set of full measure, \( \mathbb{P}(\Omega') = 1 \).

With this notion of a random fixed point \( k_*(\omega) := k_*(\theta^t \omega), t \in \mathbb{Z} \) defines a stationary solution with a time independent distribution \( \nu(B) := \mathbb{P}(k_*(\omega) \in B) = \int \chi_B(\omega) \mathbb{P}(\mathbb{d} \omega) \) of \( k_* \). It cannot be expected to observe solutions, where \( K \) and \( L \) have the same growth rates since \( k_1 \) is random. Instead, for a given noise process \( \omega \in \Omega \) we use \( k_1(\omega) \) to define a random balanced growth path to be the following random variable:

**Definition 4.1** Let \( \omega \in \Omega, K_0 \in \mathbb{R}_+, L_0 \in \mathbb{R}_+ \) and \( k_1^* \) be a stationary solution of (4.3). An orbit of \( (K_t, L_t) = (K(t, \omega, K_0, L_0), L(t, \omega, L_0)) \) of (4.1), (4.2) is called a random balanced growth path, if

\[ K_t = k_1^*(\omega)L_t \quad \text{for all } t \geq 0. \]

This definition implies the equivalence of the following three assertions:

1) \( (k_1^*(\omega)L(t, \omega, L_0), L(t, \omega, L_0)) \) is a balanced growth path for almost all \( \omega \in \Omega \).

2) \( k_* \) is a random fixed point of (4.3) satisfying (4.4).

3) \( (1 + n_0)k_1^*(\omega)(1 - \delta) - \frac{s(\xi_0, k_1^*(\omega))}{k_0^*(\omega)} \) for almost all \( \omega \in \Omega \).

In the following section we analyze conditions, where convergence in ratios to \( k_1^* \), i. e.

\[ \lim_{t \to \infty} ||k(t, \omega, k_0) - k_1^*(\omega)|| = 0 \]

implies convergence in absolute terms to a random balanced growth path as asymptote, i. e.

\[ \lim_{t \to \infty} ||K(t, \omega, K_0, L_0) - k_1^*(\omega)L(t, \omega, L_0)|| = 0 \]

or divergence in absolute terms from the asymptote to infinity, i. e.

\[ \lim_{t \to \infty} ||K(t, \omega, K_0, L_0) - k_1^*(\omega)L(t, \omega, L_0)|| = \infty. \]

**Definition 4.2** A random balanced growth path \( (K_t, L_t) = (k_1^*(\omega)L(t, \omega, L_0), L(t, \omega, L_0)) \) is called asymptotically stable, if for all initial values \( K_0 \) in a neighborhood of \( K_0 \) and \( L_0 = L_0 \) the distance \( \Delta_t = \Delta(t, \omega, K_0, L_0) := K(t, \omega, K_0, L_0) - k_1^*(\omega)L(t, \omega, L_0) \) from \( (K_t, L_t) \) to the balanced growth path \( (K_1^*, L_1^*) \) satisfies \( \lim_{t \to \infty} ||\Delta_t|| = 0. \)

\textsuperscript{11}Further on we abbreviate \( k' := \frac{\partial k}{\partial \omega} \).

\textsuperscript{12}Here, the term almost surely (a.s.) is used in a non-standard sense: a property holds \( \mathbb{P} \)-almost surely (\( \mathbb{P} \)-a.s.) if there exists a \( \theta \)-invariant set \( \Omega' \subset \Omega \) (i.e. \( \theta \Omega' = \Omega' \)) with \( \mathbb{P}(\omega') = 1 \), such that the property holds for all \( \omega \in \Omega' \).
4.1 Stability of balanced growth paths

Using these notions and concepts one can now establish similar properties of the savings function for convergence and divergence as above.

**Theorem 4.1** Let $s$ be differentiable and increasing with respect to $k$ and let $k^*_t$ be a stationary solution induced by an asymptotically stable random fixed point of

$$k_{t+1} = h(\omega_t, k_t) = \frac{(1-\delta)k_t + s(\xi_t, k_t)}{1+n_t}.$$ 

Let $[\underline{k}, \bar{k}], 0 \leq \underline{k} < \bar{k} \leq \infty$ be a positive invariant interval with $k^*_0(\omega) \in [\underline{k}, \bar{k}]$ for all $\omega \in \Omega$ and let $\inf_{\xi \times [\underline{k}, \bar{k}]} \{h'(\xi, n; k)\} > 0$.

For almost all $\omega \in \Omega$ and any $k_0 \in (\underline{k}, \bar{k})$ with $\lim_{t \to -\infty} |k(t, \omega, k_0) - k^*_t(\omega)| = 0$ (i.e. $k_0$ is in the basin of attraction of $k^*_0(\omega)$) the distance $\Delta_t := K_t - k^*_t(\omega)L_t$ to the balanced growth path $(K_t, L_t) = (k^*_t(\omega)L(t, \omega, L_0), L(t, \omega, L_0))$ satisfies

$$\lim_{t \to -\infty} |\Delta_t| = 0 \quad \text{if} \quad E \log (|h'(\omega_0, k^*_0(\omega))|) + E \log (1 + n_0) < 0, \quad (4.5)$$

$$\lim_{t \to -\infty} |\Delta_t| = \infty \quad \text{if} \quad E \log (|h'(\omega_0, k^*_0(\omega))|) + E \log (1 + n_0) > 0. \quad (4.6)$$

**Proof:**

First notice that $k_0 \neq k^*_0(\omega)$, positive invariance of $[\underline{k}, \bar{k}]$ and $\inf_{\xi \times [\underline{k}, \bar{k}]} \{h'(\xi, n; k)\} > 0$ induces $k_t \neq k^*_t(\omega)$ and hence $\Delta_t \neq 0$ for all $t \geq 0$.

From the definition $\Delta_t = K_t - k^*_t(\omega)L_t$ and by (4.1), (4.2) we obtain

$$\Delta_{t+1} = (1-\delta)K_t + s(\xi_t, k_t)L_t - k^*_{t+1}(\omega)(1+n_t)L_t$$

and hence using the solution property $k^*_{t+1}(\omega) = h(\omega_t, k^*_t(\omega))$ we get

$$\frac{\Delta_{t+1}}{\Delta_t} = \frac{(1-\delta)k_t + s(\xi_t, k_t) - k^*_{t+1}(\omega)(1+n_t)}{k_t - k^*_t(\omega)}$$

$$= \frac{(1+n_t)(h(\omega_t, k_t) - h(\omega_t, k^*_t(\omega)))}{k_t - k^*_t(\omega)}.$$ 

Since $\lim_{t \to -\infty} |k_t - k^*_t(\omega)| = 0$ we obtain for an arbitrary sufficiently small $\epsilon > 0$ and $t_0 = t_0(\epsilon, \omega) > 0$ sufficiently large that

$$\left| \frac{h(\omega_t, k_t) - h(\omega_t, k^*_t(\theta^t\omega))}{k_t - k^*_t(\omega)} - h'(\omega_t, k^*_t(\omega)) \right| < \epsilon \quad \text{for all} \quad t \geq t_0(\epsilon, \omega). \quad (4.7)$$

Note that $(\xi, n, k) \mapsto \frac{(1-\delta)k + s(\xi, k)}{1+n}$ and $(\xi, n, k) \mapsto \frac{1-\delta + s(\xi, k)}{1+n}$ are uniformly continuous on the compact set $[\underline{n}, \bar{n}] \times [\underline{n}, \bar{n}] \times [\underline{k}, \bar{k}]$.

Thus, the distance $|\Delta_t| = |K_t - k^*_t(\omega)L_t|$ to the random balanced growth path satisfies

$$(1+n_t)\left|h'(\omega_t, k^*_t(\omega))\right| |\Delta_t| < |\Delta_{t+1}| < (1+n_t)\left|h'(\omega_t, k^*_t(\omega)) + \epsilon \right| |\Delta_t|,$$

for all $t > t_0(\epsilon, \omega)$, if $\epsilon < \inf_{\xi \times [\underline{k}, \bar{k}]} \{h'(\xi, n; k)\}$. By induction we see $\Delta_t \leq |\Delta_t| < \tilde{\Delta}_t$ for all $t \geq t_0$, where $\tilde{\Delta}_t$ solves the linear random dynamical system

$$\tilde{\Delta}_{t+1} = (1+n_t)\left|h'(\omega_t, k^*_t(\omega))\right| + \epsilon \tilde{\Delta}_t, \quad \tilde{\Delta}_0 = |\Delta_0|$$
and \( \Delta_t \) solves the linear random dynamical system
\[
\Delta_{t+1} = (1 + n_t) \left( |h'(\omega_t, k_t^*(\omega))| + \varepsilon \right) \Delta_t, \quad \Delta_0 = |\Delta_0|.
\]

In the appendix we show for \( \mathbb{E} \log (h'(\omega, k_0^*(\omega))) + \mathbb{E} \log (1 + n_0) < 0 \) that the upper bound \( \Delta_t \) eventually converges to \( 0 \) \( \mathbb{P} \) -a. s. inducing the convergence result. On the other hand for \( \mathbb{E} \log (h'(\omega, k_0^*(\omega))) + \mathbb{E} \log (1 + n_0) > 0 \) we show that the lower bound \( \Delta_t \) eventually goes to infinity \( \mathbb{P} \) -a. s. implying the divergence result.

\[ \text{QED.} \]

Finally, we provide rather restrictive sufficient conditions for the convergence and divergence as analogous results to the deterministic case.

**Lemma 4.1** Let the assumptions of Theorem 4.1 be satisfied. Then

\[
\lim_{t \to \infty} |\Delta_t| = 0 \quad \text{if} \quad E_s(\xi_0, k_0^*(\omega)) < \frac{\delta}{\left( n_0 \frac{k_t^*(\omega)}{k_0^*(\omega)} + \frac{k_t^*(\omega) - k_0^*(\omega)}{k_0^*(\omega)} \right)} \quad \text{for all} \ \omega \in \Omega,
\]

\[
\lim_{t \to \infty} |\Delta_t| = \infty \quad \text{if} \quad E_s(\xi_0, k_0^*(\omega)) > \frac{\delta}{\left( n_0 \frac{k_t^*(\omega)}{k_0^*(\omega)} + \frac{k_t^*(\omega) - k_0^*(\omega)}{k_0^*(\omega)} \right)} \quad \text{for all} \ \omega \in \Omega.
\]

**Proof:**
If \( E_s(\xi_0, k_0^*(\omega)) < \frac{\delta}{\left( n_0 \frac{k_t^*(\omega)}{k_0^*(\omega)} + \frac{k_t^*(\omega) - k_0^*(\omega)}{k_0^*(\omega)} \right)} \) for all \( \omega \in \Omega \), then

\[
\mathbb{E} \log (h'(\omega, k_0^*(\omega))) + \mathbb{E} \log (1 + n_0) = \mathbb{E} \log (s'(\xi_0, k_0^*(\omega)) + (1 - \delta))
\]

\[
= \mathbb{E} \log \left( \frac{s(\xi_0, k_0^*(\omega))}{k_0^*(\omega)} E_s(\xi_0, k_0^*(\omega)) + (1 - \delta) \right)
\]

\[
= \mathbb{E} \log \left( \frac{1 + n_0}{k_0^*(\omega)} - (1 - \delta) \right) E_s(\xi_0, k_0^*(\omega)) + (1 - \delta)
\]

\[
< \mathbb{E} \log \left( \frac{1 + n_0}{k_0^*(\omega)} - (1 - \delta) \right) \frac{\delta}{\left( n_0 \frac{k_t^*(\omega)}{k_0^*(\omega)} + \frac{k_t^*(\omega) - k_0^*(\omega)}{k_0^*(\omega)} \right)} + (1 - \delta)
\]

\[
= \mathbb{E} \log(1) = 0.
\]

The same arguments can be used for divergence when the reverse inequality “>” holds. \( \text{QED.} \)

Since it is difficult to verify such a condition for all \( \omega \in \Omega \) we provide the two more restrictive sufficient conditions on the elasticity for convergence. The first one

\[
\lim_{t \to \infty} |\Delta_t| = 0 \quad \text{if} \quad \sup_{[\xi_{\min}, \xi_{\max}] \times [k, k]} \frac{E_s(\xi, k)}{\left( 1 + n_{\max} \right) \frac{k - 1}{k} + \delta}
\]

describes an upper bound for the elasticity of the savings function which depends only on features of the support of the noise process. The second one

\[
\lim_{t \to \infty} |\Delta_t| = 0 \quad \text{if} \quad E_s(\xi_0, k_0^*(\omega)) < \frac{\delta}{\mathbb{E} \left( n_0 \frac{k_t^*(\omega)}{k_0^*(\omega)} + \frac{k_t^*(\omega) - k_0^*(\omega)}{k_0^*(\omega)} \right) + \delta} \quad \text{for} \ \omega \in \Omega.
\]
is less restrictive but more difficult to verify, and it may depend on initial the conditions \((n_0, k_0)\). Its validity can be seen from

\[
\mathbb{E} \log \left( \left(1 + n_0 \frac{k_1^*(\omega)}{k_0^*(\omega)} - (1 - \delta) \right) E_s(\xi_0, k_0^*(\omega)) + (1 - \delta) \right) \\
\leq \mathbb{E} \left( \left(1 + n_0 \frac{k_1^*(\omega)}{k_0^*(\omega)} - (1 - \delta) \right) E_s(\xi_0, k_0^*(\omega)) - \delta \right) \\
< \mathbb{E} \left( \left(1 + n_0 \frac{k_1^*(\omega)}{k_0^*(\omega)} - (1 - \delta) \right) \frac{\delta}{\mathbb{E} \left( n_0 \frac{k_1^*(\omega)}{k_0^*(\omega)} + \frac{k_1^*(\omega) - k_0^*(\omega)}{k_0^*(\omega)} \right)} + \delta - \delta \right) \\
= 0.
\]

These properties show that the results from the deterministic case carry over to the stochastic case when the perturbations are sufficiently small.

### 4.2 Numerical simulations

![Graphs](a) \(\beta = 0.4 < \frac{\delta}{3} = \frac{\delta}{n+\delta}\)  

![Graphs](b) \(\beta = 0.8 > \frac{\delta}{3} = \frac{\delta}{n+\delta}\)  

![Graphs](c) \(\beta = 0.4 < \frac{\delta}{3} = \frac{\delta}{n+\delta}\)  

![Graphs](d) \(\beta = 0.8 > \frac{\delta}{3} = \frac{\delta}{n+\delta}\)

**Figure 9:** Convergence and Divergence of the capital intensity \(k_t\) and of the distance \(\Delta_t\)

We illustrate the results for the stochastic case using the Solow–model with a constant savings propensity, a Cobb–Douglas production function with constant production elasticity \(\alpha\), and a multiplicative Hicks neutral technological shock.
The random perturbations are modeled as an i.i.d. process of the form $n_t = n(1 + 2 \cdot X_t) \sim [-0.01, 0.03]$, and $\xi_t = \xi(1 + 0.5 \cdot X_t) \sim [0.5, 1.5]$, where all $X_i \sim [-1, 1]$ are uniformly distributed. We assume $s(\xi, k) = \xi \bar{s} f(k) = \xi \bar{s} A_k^\alpha$, where $E_s(\xi, k) = E_f(k) = \alpha$ is constant. Moreover, we choose $A = \frac{n + \delta}{\bar{s}}$, such that the fixed point of the deterministic model with $n, \xi = 1$ is normalized to $\bar{k} = 1$ for all parameter values $\alpha$. All simulations are carried out using MACRODYN, a software package for discrete time dynamical systems (see Böhm 2003)).

Figure 9 displays the time series of $k_t$ and $\Delta_t$ of the intensity form

$$k_{t+1} = \frac{(1 - \delta)k_t + \xi_t \bar{s} A_k^\alpha}{1 + n_t}$$

with parameters $\bar{s} = \frac{1}{2}$, $n = 0.01$, $\delta = 0.02$, $A = \frac{n + \delta}{\bar{s}} = 0.06$, for the same noise path, and for multiple initial values $K_0 = 0.001$, $0.1$, $0.5$, $1.0$, $1.5$, $2.0$, $10.0$, $20.0$ and $L_0 = 1$. Panels (a) and (c) correspond to the situation with $\alpha = 0.4 < \frac{2}{3} = \frac{\delta}{n + \theta}$ which induces a stable balanced growth path in $(K; L)$–space. Thus, one observes the point wise convergence of both variables $k$ and $\Delta$ for arbitrary initial conditions. In contrast, panels (b) and (d) display the results for $\alpha = 0.8 > \frac{2}{3} = \frac{\delta}{n + \theta}$, which induces unstable balanced growth paths in $(K; L)$–space. Notice that the speed of convergence of the capital intensities to the random fixed point is much slower than in panels (a), (c), while the distance $\Delta$ diverges.

Figure 10: Convergence and Divergence in $(L; K)$–space
We show:

for sufficiently small $\epsilon > 0$, $\mathbb{E} \log (h'(\omega_0, k_0^* (\omega)) + \epsilon) > 0$ implies $\mathbb{E} \log (\tilde{M}(\omega)) > 0$.

We show:

for sufficiently small $\epsilon > 0$, $\mathbb{E} \log (h'(\omega_0, k_0^* (\omega)) + \epsilon) > 0$ and $\bar{\epsilon} > 0$ with $\epsilon < \bar{\epsilon} \inf_{Z \times \{\omega \}} \max \{h'(\xi, n, k)\}$, where $\bar{\epsilon} = e^{-r} - 1 > 0$. Then, $\frac{\epsilon}{\log (h'(\omega_0, k_0^* (\omega)))} < \bar{\epsilon}$ and $\tilde{\epsilon} = e^{-r} - 1 > 1$ implies

$$
\mathbb{E} \log (|h'(\omega_0, k_0^* (\omega))| + \bar{\epsilon}) = \mathbb{E} \log \left( |h'(\omega_0, k_0^* (\omega))| \left( 1 + \frac{\epsilon}{|h'(\omega_0, k_0^* (\omega))|} \right) \right)
$$

$$
< \mathbb{E} \log \left( |h'(\omega_0, k_0^* (\omega))| (1 + \bar{\epsilon}) \right)
$$

$$
= \mathbb{E} \log \left( |h'(\omega_0, k_0^* (\omega))| \right) + \log (1 + \bar{\epsilon})
$$

$$
= \mathbb{E} \log \left( |h'(\omega_0, k_0^* (\omega))| \right) - r
$$

$$
= -\mathbb{E} \log (1 + n_0).
$$

and $\mathbb{E} \log (\tilde{M}(\omega)) = \mathbb{E} \log (h'(\omega_0, k_0^* (\omega)) + \epsilon) + \mathbb{E} \log (1 + n_0) < 0$. Hence, the linear random dynamical system $\Delta_{t+1} = \tilde{M}(\theta^t \omega) \Delta_t$, $\Delta_0 = |\Delta_0|$ has a unique random fixed point $\mathcal{O}(\omega) \equiv 0$, which is asymptotically stable. Thus, $|\Delta_t|$ is bounded above by $\bar{\Delta}_t$ satisfying $\lim_{t \to \infty} \bar{\Delta}_t = 0 \mathbb{P}$ -a. s. implying $\lim_{t \to \infty} |\Delta_t| = 0 \mathbb{P}$ -a. s.

Second, for $\nu := \mathbb{E} \log (h'(\omega_0, k_0^* (\omega))) + \mathbb{E} \log (1 + n_0) > 0$ we show $\mathbb{E} \log (\tilde{M}(\omega)) > 0$, $\tilde{M}(\omega) := (1 + n_0) (h'(\omega_0, k_0^* (\omega)) - \epsilon)$ for sufficiently small $\epsilon > 0$. Let $\epsilon > 0$ with $\epsilon < \bar{\epsilon} \inf_{Z \times \{\omega \}} \max \{h'(\xi, n, k)\}$,
where \( \tilde{e} := 1 - e^{-r} \in (0, 1) \). Then \( -\frac{h'(\omega_0, k_0^*(\omega))}{h'(\omega_1, k_0^*(\omega))} > -\tilde{e} > -1 \) implies

\[
\mathbb{E} \log \left( |h'(\omega_0, k_0^*(\omega))| - \epsilon \right) = \mathbb{E} \log \left( |h'(\omega_0, k_0^*(\omega))| \left( 1 - \frac{\epsilon}{|h'(\omega_0, k_0^*(\omega))|} \right) \right) > \mathbb{E} \log \left( |h'(\omega_0, k_0^*(\omega))| (1 - \epsilon) \right) = \mathbb{E} \log \left( |h'(\omega_0, k_0^*(\omega))| \right) + \log (1 - \epsilon)
\]

\[
= \mathbb{E} \log \left( |h'(\omega_0, k_0^*(\omega))| \right) - r = -\mathbb{E} \log (1 + n_0)
\]

and hence \( \mathbb{E} \log \left( \frac{M(\omega)}{\omega_0} \right) = \mathbb{E} \log \left( |h'(\omega_0, k_0^*(\omega))| - \epsilon \right) + \mathbb{E} \log (1 + n_0) > 0. \)

If, in addition, \( \epsilon < \inf_{\xi \in \mathbb{R}} h'(\xi, n, k) \) we see \( M(\omega) > 0 \) and hence \( M_{t_0} \neq 0 \) implies \( \Delta_{t} \neq 0 \) for all \( t \geq t_0 \), such that \( Q_t := \frac{1}{\Delta_{t}} \) solves

\[
Q_{t+1} = \frac{1}{M(\theta_t, \omega)} Q_t, \quad Q_{t_0} = \frac{1}{\Delta_{t_0}}.
\]

Therefore, \( O(\omega) \equiv 0 \) is a unique random fixed point of \( Q_{t+1} = \frac{1}{M(\theta_t, \omega)} Q_t \) which is asymptotically stable since \( \mathbb{E} \log \left( \frac{1}{M(\omega)} \right) = -\mathbb{E} \log \left( \frac{M(\omega)}{\omega} \right) < 0. \) Thus, \( \lim_{t \to \infty} Q_t = 0 \) \( \mathbb{P} \)-a.s. implies \( \lim_{t \to \infty} \Delta_{t} = \infty \) \( \mathbb{P} \)-a.s. and we get the divergence result.

References


